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Multidimensional inversion formalism as a compatibility condition between different linear differential systems

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Abstract. We consider, as in a previous paper, multidimensional inversion-like integral equations (IEs)—from which we can construct a class of potentials without introducing the data—associated with a system of n-linear first-order partial differential equations in R^n. Firstly we emphasise that, contrary to the one-dimensional case, there exists different IEs associated with the same system in R^n. Then we study the properties of the reconstructe potentials in a case where the scalar kernels of the IE depend in fact upon only two independent variables, and we find that: (i) there exists a general method for constructing real, confined potentials in R^n; (ii) for $n \geqslant 3$ the potentials satisfy well-defined nonlinear constraints. For $n = 3$ the IE is common to two different linear differential systems, for $n = 4$, to three systems, and so on. The compatibility conditions between these linear systems reproduce the above nonlinear constraints (for instance, for $n = 3$ we get the nonlinear three-wave equations in two spatial coordinates); (iii) combining (i) and (ii) we provide explicit examples for a new class of nonlinear equations reducible to the inversion formalism and which have confined solutions in R^n.

1. Introduction

Currently there is great interest in the explicit construction of simple multidimensional solutions of nonlinear equations which are real, confined in R^n. Very few examples are known to work: for instance, the one-instanton solution for the Yang–Mills equations in R^4 (Belavin *et al* 1975), the two-spatial-dimensional KDK equation (Manakov *et al* 1977), the two-spatial-dimensional nonlinear three-wave equations (Cornille 1978a), and the generalised two-spatial-dimensional nonlinear Schrödinger equation (Cornille 1978b). For those nonlinear equations which are reducible to the inversion formalism, an essential preliminary problem is the existence (or otherwise) of reconstructed potentials which are confined in R^n. This means that we must necessarily investigate carefully the properties of the inversion-like integral equations (IEs), from which we can construct a class of potentials associated to linear systems (without introducing the data).

In the preceding paper (Cornille 1978a) we built IEs associated to linear first-order differential systems in R^n, and in the present one we extend our study of the properties of the IEs. We recall briefly the previous results:

Let us consider the $n \times n$ differential linear system

$$(L_{\mathrm{I}} + \mathrm{i}k\Lambda - Q_{\mathrm{I}})\psi = 0, \tag{1}$$

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where L_{I} and Λ are diagonal, $\Lambda = (\delta_{ij}\lambda_i)$, $L_{\mathrm{I}} = (\delta_{ij}\partial/\partial_{x_i})$; Q_{I} is an $n \times n$ potential,

$$Q_{\mathrm{I}} = \begin{pmatrix} q_1^1 & \dots & q_1^n \\ \vdots & & \\ q_n^1 & \dots & q_n^n \end{pmatrix};$$

and ψ is a column vector. Assuming a representation of the solutions of (1) for a set $\{\psi_j\}$ of n functions

$$\psi_j = \left(u^0_{\lambda_i}(x_j)\delta_{ij} + \int_{x_j}^{\infty} K^l_i(x_1, \dots, x_n; y) u^0_{\lambda_l}(y)\,\mathrm{d}y \right), \qquad \left(\frac{\partial}{\partial x} + \mathrm{i}\lambda_j k\right) u^0_{\lambda_j}(x) = 0 \tag{2a}$$

and substituting into $(2a)$ we obtained

$$\left(\frac{\partial}{\partial x_j} + \frac{\lambda_j}{\lambda_i}\frac{\partial}{\partial y}\right) K^i_j = \sum_{m \neq j} K^i_m \hat{K}^m_j \lambda_j (\lambda_m)^{-1}, \qquad \hat{K}^m_j = K^m_j(y = x_m),$$
$$q^i_i = 0, \qquad q^i_j = \lambda_j(\lambda_i)^{-1}\hat{K}^i_j. \tag{3a}$$

In this paper we call *formalism* 'a' all the properties deduced from the input representation $(2a)$, and we will compare with other formalisms coming from different input representations of (1). If we try to find an IE of equation (1) which when written in matrix notation is of the type

$$\mathcal{K}(x_1, \dots, x_n; y) = \tilde{\mathcal{F}}(x_1, \dots, x_n; y) + \int_{-\infty}^{+\infty} \mathcal{F}(s; x_1, \dots, x_n; y)\mathcal{K}(x_1, \dots, x_n; s)\,\mathrm{d}s,$$
$$\mathcal{K} = (K^i_j), \qquad \mathcal{F} = (F^i_j \theta(s - x_j)), \qquad \tilde{\mathcal{F}} = (\tilde{F}^i_j = F^i_j(s = x_j)), \tag{4}$$

and such that the K^i_j (solutions of (4)) satisfy the nonlinear equations $(3a)$, then we find that the scalar kernels F^i_j must satisfy

$$\left(\lambda_m^{-1}\frac{\partial}{\partial x_m} + \lambda_j^{-1}\frac{\partial}{\partial s} + \lambda_i^{-1}\frac{\partial}{\partial y}\right) F^i_j = 0, \qquad m = 1, 2, \dots, n$$

or $\hfill (4a)$

$$F^i_j = F^i_j\left(u^i_j = \lambda_j(x_j - s) + \sum_{l \neq j} \lambda_l x_l;\ v^i_j = \lambda_i(x_i - y) + \sum_{l \neq i} \lambda_l x_l \right),$$

and of course well-defined boundary conditions. The fact that the F^i_j given by $(4a)$ depend upon two independent variables has the following consequences: (i) for $n = 2$ the potentials can be confined in R^2 (for instance, there exist confined solutions of the generalised nonlinear Schrödinger equation (Cornille 1978b)); (ii) for $n \geqslant 3$ the potentials reconstructed from (4) and $(4a)$ are not confined and have to satisfy well-defined nonlinear constraints—the nonlinear three-wave equation

$$\left(\lambda_k \lambda_j^{-1}\frac{\partial}{\partial x_j} + \lambda_k \lambda_i^{-1}\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i}\right) q^i_j = 2 q^i_k q^k_j \tag{5a}$$

for $n = 3$, and generalisations of it for $n > 3$, for instance for $n = 4$

$$\left(\frac{1}{\lambda_k}\frac{\partial}{\partial x_k} - \frac{1}{\lambda_l}\frac{\partial}{\partial x_l}\right) q^i_j = \frac{1}{\lambda_l} q^i_l q^l_j - \frac{1}{\lambda_k} q^i_k q^k_j, \qquad i, j, l, k \text{ all different}$$
$$\left(\frac{1}{\lambda_j}\frac{\partial}{\partial x_j} + \frac{1}{\lambda_i}\frac{\partial}{\partial x_i} - \frac{1}{\lambda_k}\frac{\partial}{\partial x_k}\right) q^i_j = \frac{2}{\lambda_k} q^i_k q^k_j + \frac{1}{\lambda_l} q^i_l q^l_j. \tag{6a}$$

In this paper we pursue our investigations of the IE associated to (1); (i) the non-uniqueness of the IE, or the possibility of introducing other formalisms starting from different input representations of ψ; (ii) the possibility of confined solutions in R^n, even with the scalar kernels of the IE depending upon two different variables; (iii) the fact that the same IE can be associated with different linear systems for which compatibility conditions lead to nonlinear constraints for the potentials like ($5a$) or ($6a$).

2. Problems under investigation

2.1. *Non-uniqueness of the possible representations of the solutions associated to linear systems and non-uniqueness of the* IE *in the multidimensional case*

There exist two possible generalisations of ($2a$) associated to (1), and consequently there exist formalisms other than the one called a in the introduction.

(i) Firstly let us consider what we shall call *formalism 'b'*,

$$\psi_J=\left(u_j(x_1,\ldots,x_n)\delta_{ij}+\int_{x_i}K^j_i(x_1,\ldots,x_n;y)u_j(x_1,\ldots,x_{j-1},y,x_{j+1},\ldots,x_n)\,\mathrm{d}y\right),$$

$$u_j=u^0_{\lambda_j}(x_j)v^0_j(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n),\qquad\left(\frac{\partial}{\partial x}+\mathrm{i}\lambda k\right)u^0_\lambda(x)=0,\tag{2b}$$

where v^0_j is an arbitrary function. This arbitrariness is due to the fact that if we consider a set ψ^0_j (solutions of (1) when $Q_\mathrm{I}\equiv0$), $(L_\mathrm{I}+\mathrm{i}\Lambda)(\psi^0_1,\ldots,\psi^0_n)\equiv0$, we can take either $\psi^0_j=(u^0_{\lambda_j}(x_j)\delta_{ij})$ as in ($2a$) or $\psi^0=(u^0_{\lambda_j}(x_j)v^0_j(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n)\delta_{ij})$ as in ($2b$). In § 3, taking into account this freedom in the representations of ψ, we shall deduce, for a particular choice of $\{v^0_j\}$, a corresponding IE ($4b$) which will be a generalisation of ($4a$) because ($2b$) reduces to ($2a$) when $v^0_j=1$. We introduce arbitrary constants λ^i_j, $i\neq j$, define $\bar\lambda^i_j=(\lambda_i-\lambda^j_i)\lambda_j^{-1}$ for $i\neq j$, $\bar\lambda^i_i=1$, and choose $v^0_j=\Pi_{l\neq j}u^0_{\lambda^l_l}(x_l)$ in ($2b$). In this way we shall obtain an IE with n^2-n parameters which we can introduce into the formalism although not being present in (1). Seeking an IE of the equation (4) type, we obtain for the kernels

$$\left(\frac{\partial}{\partial x_m}+\bar\lambda^i_m\frac{\partial}{\partial y}+\bar\lambda^j_m\frac{\partial}{\partial s}\right)F^i_j=0,\qquad m=1,2,\ldots,n$$

or ($4b$)

$$F^i_j=F^i_j\left(u^i_j=x_j-s+\sum_{m\neq j}\bar\lambda^j_m x_m;\,v^i_j=x_i-y+\sum_{m\neq i}\bar\lambda^i_m x_m\right),$$

whereas the solutions are linked to the potentials following $q^i_j=\bar\lambda^i_j\hat K^i_j$, $q^i_i=0$. This new formalism b generalises the previous one a, because when $\lambda^i_j\to0$, then $u^0_{\lambda^i_j}\to$ constant, and ($2b$) is reduced to ($2a$).

In this new formalism we can consider the case where some $q^i_j\equiv0$ by choosing $\lambda^i_j=\lambda_j$ or $\bar\lambda^i_j=a$ without modifying the parameters of equation (1) (contrary to formalism a for which $q^i_j\equiv0$ requires $\lambda_j=0$). This freedom in the formalisms (we shall call it bb) will be useful in § 4, when we shall consider the possibility of confined solutions in R^n. However, let us notice that the kernels F^i_j in ($4b$) still depend upon two independent variables.

(ii) Secondly we define *formalism* '*c*'. Instead of the representation $\int_{x_j} K^i_j u_j \, dy$ in (2*b*), where the integration path is linked with only one coordinate x_j, we could take a superposition of such terms where all coordinates play a similar role:

$$\psi_J = \Big(u_j(x_1, \ldots, x_j, \ldots, x_n)\delta_{ij} + \sum_m \int_{x_m} K^i_{i,m}(x_1, \ldots, x_n; y) u_j(x_1, \ldots, x_{m-1}, y, x_{m+1}, \ldots, x_n) \, dy \Big). \qquad (2c)$$

Although for simplicity we shall not derive in the general case the corresponding IE, this could be done. Here we report only the results for the case $n = 2$. In the Zakharov–Shabat (1974) multidimensional theory, only one coordinate is really a variable, the others being parameters. This would correspond in (2*a*, *b*, *c*) to always choosing the same coordinate at the end of the integration path. In conclusion we can enlarge the class of representations of the ψ_j; however, seeking an IE of the equation (4) type in the cases that we have considered, we always find the feature that for $n > 3$ the number of really independent variables for the scalar kernels of the IE is less than n. Consequently we expect to meet difficulties concerning the confinement properties of the $\{q^i_j\}$.

2.2. Confined solutions in R^n

In § 4 we provide a general procedure for the confinement properties of the reconstructed potentials in R^n. Let us iterate the IE of the equation (4) type with (4*a*) or (4*b*):

$$\mathcal{K} = \sum_{n=1}^{n=\infty} x\mathcal{K}_n, \qquad \mathcal{K}_1 = \tilde{\mathcal{F}}, \qquad \mathcal{K}_2 = \int \mathcal{F}\tilde{\mathcal{F}}, \qquad \mathcal{K}_3 = \iint \mathcal{F}\mathcal{F}\tilde{\mathcal{F}}, \qquad \ldots . \qquad (4')$$

Owing to the fact that these scalar kernels depend only upon two independent variables, the terms present in the matrix $\mathcal{F}$ cannot be confined in more than R^2, those present in $\int \mathcal{F}\tilde{\mathcal{F}}$ in more than R^3, those present in $\iint \mathcal{F}\mathcal{F}\tilde{\mathcal{F}}$ in more than R^4, and so on. Thus, in order to have confined potentials in R^3, it is necessary that the potentials q^i_j with the same indices as the $\tilde{F}^i_j \neq 0$ be put to zero, otherwise they are present in the first term $\tilde{\mathcal{F}}$ of the iteration (4′). If we use formalism *a* this requires $\lambda_j \equiv 0$, whereas if we consider formalism *b* we only put $\tilde{\lambda}^i_j = 0$ without modifying system (1). So we take advantage of the freedom due to the *ansatz* (2*b*) in such a way that only the $\tilde{F}^i_j \neq 0$ have corresponding q^i_j potentials identically zero, whereas the $q^i_j \not\equiv 0$ appear firstly in the second iteration $\int \mathcal{F}\tilde{\mathcal{F}}$ and can be confined in R^3. Similarly if the only non-vanishing q^i_j appear firstly in the third term $\iint \mathcal{F}\mathcal{F}\tilde{\mathcal{F}}$, then these potentials can be confined in R^4, and so on for the building of a class of confined potentials in R^n. In order to illustrate these possibilities we give now a simple explicit example where the kernels are degenerate, $F^i_j = g^i_j(u^i_j(s))h^i_j(v^i_j(y))$ and

$$\mathcal{F} = \begin{pmatrix} 0 & F^1_2 & 0 \\ 0 & 0 & F^2_3 \\ F^3_1 & 0 & 0 \end{pmatrix}. \qquad (7)$$

(i) We consider formalism *a* with $F^i_j = g^i_j(\lambda_j(x_j - s) + \bar{x}_i + \bar{x}_k)h^i_j(\lambda_i(x_i - h) + \bar{x}_j + \bar{x}_k)$, $\bar{x}_m = \lambda_m x_m$, $k \neq j$ and $k \neq i$, $q^i_j = \lambda_j(\lambda_i)^{-1}\hat{K}^i_j$, and obtain, for the behaviour of the q_i in R^3, two classes depending upon the corresponding F^i_j being identically zero or not:

$$D\hat{K}^i_j = g^i_j(\bar{x}_i+\bar{x}_k)h^i_j(\bar{x}_j+\bar{x}_k), \qquad (i,j)=(1,2),(2,3),(3,1), \qquad (7a_1)$$

$$D\hat{K}^i_j = h^i_k(\bar{x}_j+\bar{x}_k)g^k_j(\bar{x}_i+\bar{x}_k)A^i_{j,k}(\bar{x}_i+\bar{x}_j), \qquad (i,j)=(2,1),(3,2),(1,3), \qquad (7a_2)$$

with

$$A^i_{j,k}(x)=\lambda_k^{-1}\int_{-x}^{\infty} g^i_k(-u)h^k_j(-u)\,\mathrm{d}u, \qquad D=1-A^3_{21}(\bar{x}_3+\bar{x}_2)A^1_{32}(\bar{x}_1+\bar{x}_3)A^2_{13}(\bar{x}_2+\bar{x}_1).$$

Let us assume for the functions g^i_j and h^i_j

$$\lim_{|u|\to\infty} g^i_j(u)=0, \qquad \lim_{|v|\to\infty} h^i_j(v)=0, \qquad \lim_{|x|\to\infty} A^i_{j,k}(x)=0$$

or

$$\int_{-\infty}^{+\infty} g^i_k(u)h^k_j(u)\,\mathrm{d}u=0, \qquad (8)$$

and that $|A^i_{j,k}(x)|$ is bounded. It follows that the q^i_j belonging to the second class $(7a_2)$ are confined in R^3, whereas those of the first class $(7a_1)$ are not. We remark that the $\hat{K}^i_j$ of the second class appear firstly in the second iteration of $\mathscr{K}$, whereas those of the first class are present in the first term $\tilde{\mathscr{F}}$. Wanting to put to zero the q^i_j belonging to the first class we see that $\lambda_1=\lambda_2=\lambda_3=0$, leading also to the vanishing of the q^i_j belonging to the second class, and this is impossible.

(ii) We consider formalism bb: for the same $\mathscr{F}$ (equation (7)) as above we assume $\tilde{\lambda}^1_2=\tilde{\lambda}^2_3=\tilde{\lambda}^3_1=0$ or $q^1_2=q^2_3=q^3_1=0$, so that the degenerate kernels of the $(4b)$ type are reduced to $F^i_j=g^i_j(x_j-s+\tilde{\lambda}^i_jx_i)h^i_j(x_i-y+\tilde{\lambda}^i_kx_k)$ for $(i,j)=(1,2),\ (2,3),\ (3,1)$. We obtain for the only three $q^i_j=\tilde{\lambda}^i_j\hat{K}^i_j\neq 0$

$$D\hat{K}^i_j = h^i_k(\tilde{\lambda}^i_jx_j)g^k_j(\tilde{\lambda}^i_kx_k)A^i_{j,k}(\tilde{\lambda}^k_ix_i), \qquad (i,j)=(2,1),(3,2),(1,3), \qquad (7bb)$$

where now

$$A^i_{j,k}(x)=\int_{-x}^{\infty} g^i_k(-u)h^k_j(-u)\,\mathrm{d}u, \qquad D=1-A^3_{21}(\tilde{\lambda}^1_3x_3)A^1_{32}(\tilde{\lambda}^2_1x_1)A^2_{13}(\tilde{\lambda}^3_2x_2).$$

If the $\{g^i_j, h^i_j\}$ verify assumption (8), then the potentials are confined in R^3 (the Fredholm determinants D are bounded, and we exclude in our discussion the cases where D vanishes).

2.3. The fact that our IEs *for $n\geqslant 3$ represent a compatibility condition between different linear systems is studied in* § 5

In fact the q^i_j reconstructed either from formalism a or formalism b satisfy constraints for $n\geqslant 3$; they are solutions of well-defined nonlinear equations like $(5a)$ and $(6a)$. In § 5 we explain the origin of these constraints.

For $n=3$ we show that there exists another linear system

$$(L_{\mathrm{II}}-Q_{\mathrm{II}})\psi=0 \qquad (1')$$

associated to (1), with the same solution ψ as given by $(2a)$ or $(2b)$ and leading to the same IE $(4a)$ or $(4b)$. From the compability condition

$$[L_{\mathrm{I}}\quad, L_{\mathrm{II}}\quad]\psi=0 \qquad (9)$$

we obtain for the scalar potentials q_j^i the above nonlinear constraints—for instance (5*a*) for the IE (4*a*).

For $n=4$ the reconstructed solution ψ from the IE (4*a*) is also a solution of two other differential linear systems (1′) and (1″), $(L_{\rm III}-Q_{\rm III})\psi=0$, so that the compatibility conditions between (1), (1′), (1″) give the constraints that the $\{q_j^i\}$ must satisfy—for instance (6*a*) if we start with (4*a*). In this way if we start from formalism *bb* in such a way that only potentials confined in R^n are present (by requiring some $\bar\lambda_j^i=0$) we can exhibit explicit examples of nonlinear equations having confined solutions in R^n. For instance, for $n=3$ the nonlinear integro-differential equations

$$\frac{\partial}{\partial x_i}q_j^i(x_i,x_j,x_k)=\int_{x_k}^{\infty}q_j^iq_k^i\,\mathrm{d}x_k'\int_{x_j}^{\infty}q_j^iq_i^k\,\mathrm{d}x_j', \qquad (i,j)=(2,1),(3,2),(1,3) \tag{5bb}$$

have confined solutions in R^3—for instance (7*bb*)—and there exists a generalisation of this kind for $n>3$.

This intriguing property that our IE of the (4*ab*) type is common to different linear systems is due to the fact that different linear differential operators put to zero the scalar kernels $\tilde F_j^i$. Consider, for instance, $n=3$ and formalism *a*:

$$\left(\frac{\partial}{\partial x_j}+\lambda_j\lambda_i^{-1}\frac{\partial}{\partial y}\right)\tilde F_j^i=0$$

is a necessary relation for the IE (4*a*) to be linked to system (1); however, there exists another relation

$$\left(\lambda_q^{-1}\frac{\partial}{\partial x_q}-\lambda_k^{-1}\frac{\partial}{\partial x_k}\right)\tilde F_j^i=0, \qquad q\neq j,\ k\neq j$$

which leads to system (1′). Similarly for formalism *b*:

$$\left(\frac{\partial}{\partial x_j}+\bar\lambda_j^i\frac{\partial}{\partial y}\right)\tilde F_j^i=0$$

leads to (1), whereas

$$\left[(\bar\lambda_i^j)^{-1}\frac{\partial}{\partial x_i}-(\bar\lambda_k^j)^{-1}\frac{\partial}{\partial x_k}+\frac{\partial}{\partial y}\left(\frac{1}{\bar\lambda_i^j}-\frac{\bar\lambda_k^i}{\bar\lambda_k^j}\right)\right]\tilde F_j^i=0$$

connects the IE (4*b*) to another system (1′).

2.4. *In the following sections we shall derive IEs associated to linear differential systems*

We sketch here very briefly the general method (Cornille 1978b) and as an illustration we consider a second-order case, emphasising the possibility of different IEs in the multidimensional case.

(i) Let L be a multidimensional linear differential operator with a constant or eigenvalue term, Q a 'potential', ψ a solution,

$$(L-Q)\psi=0, \qquad L\psi_0=0, \tag{I}$$

and ψ_0 is known. The aim is to define a formalism from which we can construct both a class of ψ and Q.

(ii) We postulate a representation of ψ,

$$\psi = \psi_0 + \mathscr{L}(\psi_0 \mathscr{K}), \tag{II}$$

where $\mathscr{L}$ is some functional integral and $\mathscr{K}$ the transform of ψ.

(iii) We put (II) into (I) and obtain that $\mathscr{K}$ both satisfies well-defined nonlinear equations (III) and is linked to Q(III).

(iv) We seek both an integral equation of the type

$$\mathscr{K} = \tilde{\mathscr{F}} + \int \mathscr{F}\mathscr{K} \qquad (\mathscr{F} \text{ linked to } \tilde{\mathscr{F}}) \tag{IV}$$

and the properties that $\mathscr{F}$ has to satisfy in order that $\mathscr{K}$, the solution of (IV), verifies the nonlinear equations (III). We find that $\mathscr{F}$ satisfies a linear equation that is completely integrable and is linked to the linear part of (III). We construct a class of $\mathscr{F}$ leading to a class of $\mathscr{K}$ and consequently to a class of ψ and Q. We have to take into account the boundary conditions occuring in the formalism.

As an application we consider a scalar second-order example in R^n,

$$(\Delta_n + k^2 - V)\psi = 0, \qquad (\Delta_n + k^2)\psi_0(x_1, \ldots, x_n) = 0, \qquad \Delta_n = \sum_1^n \frac{\partial^2}{\partial x_i^2}, \tag{I}$$

$$\psi = \psi_0(x_1, \ldots, x_n) + \sum_{i=1}^n \alpha_i \int_{x_i}^\infty K(x_1, \ldots, x_n; y)\psi_0(x_1, \ldots, x_{i-1}, y, y_{i+1}, \ldots, x_n)\, dy, \tag{II}$$

where the α_i are arbitrary constants. We put (II) into (I), and assuming boundary conditions we obtain

$$\begin{gathered} \sum_i \alpha_i \int_{x_i}^\infty \psi_0\Big(\Box_n + 2\sum_j \alpha_j \hat{K}_{j,x_j}\Big)K\, dy - \psi\Big(V + 2\sum_i \alpha_i \hat{K}_{i,x_i}\Big) = 0, \\ \Box_n = \Delta_n - \frac{\partial^2}{\partial y^2}, \qquad \hat{K}_i = K(x_1, \ldots, x_n; y = x_i), \qquad \hat{K}_{i,x_i} = \frac{\partial}{\partial x_i}\hat{K}_i, \end{gathered} \tag{III}$$

if

$$\lim_{y\to\infty} K\frac{\partial \psi_0}{\partial y}(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) = 0,$$

$$\lim_{y\to\infty} \psi_0(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)\frac{\partial K}{\partial y} = 0.$$

The identity is satisfied if

$$\Big(\Box_n + 2\sum_i \alpha_i \hat{K}_{i,x_i}\Big)K \equiv 0, \qquad V + 2\sum_i \alpha_i \hat{K}_{i,x_i} = 0. \tag{III}$$

Let us consider the following IE assuming that the solution exists and is unique:

$$\begin{aligned} K(x_1, \ldots, x_n; y) &= F(x_1, \ldots, x_n; y) + \sum_i \alpha_i \int_{x_i}^\infty F(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_n; y) \\ &\quad \times K(x_1, \ldots, x_n, s)\, ds, \qquad \Box_n F_n = 0. \end{aligned} \tag{IV}$$

If we assume the boundary conditions

$$\lim_{s\to\infty} K(x_1, \ldots, x_n; s)\frac{\partial F}{\partial s}(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_n; y) = 0,$$

$$\lim_{s\to\infty} F(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_n; y)\frac{\partial K}{\partial s}(x_1, \ldots, x_n; s) = 0,$$

then K, the solution of (IV), satisfies the nonlinear equations (III). (For the proof we apply $\Box_n$ to both sides of (IV), obtain

$$\Box_n K = -2F\sum_j \alpha_j \hat{K}_{j,x_j} + \sum_i \alpha_i \int_{x_i} F\Box_n K,$$

and compare with the solution of (IV).) Let us assume now that α_i takes only the values 0 or 1. In the one-dimensional case we have only one possibility, whereas for $n > 1$ we obtain $n^2 - 1$ different representations (II) and the IE (IV). Let us also remark that F has $n+1$ variables, $x_1, \ldots, x_n$, y, and only one constraint, $\Box_n F = 0$. On the other hand for the IE associated to system (1) that we build in the following section, the number of variables of the scalar kernels F^i_j minus the number of constraints will always be 2.

3. Nonuniqueness of the inversion-like integral equations associated to the (1)

As was previously explained, the non-uniqueness considered here comes from the fact that we can start with different representations of the solutions of (1). We give the results obtained with formalism b and restrict the study of formalism c to the $n = 2$ case.

3.1. *Formalism b*

(i) We start with equation (1), $(L_{\mathrm{I}} + \mathrm{i}k\Lambda - Q_{\mathrm{I}})\psi = 0$, and rewrite it with scalar quantities and

$$\psi = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}:$$

$$\left(\frac{\partial}{\partial x_i} + \mathrm{i}k\lambda_i\right)u_i - \sum_{l=1}^{n} q^l_i u_l = 0. \tag{1}$$

We introduce $n^2 - n$ arbitrary constants λ^i_j, with $i \neq j$, and define $\bar{\lambda}^i_j = (\lambda_j - \lambda^i_j)\lambda_i^{-1}$ for $i \neq j$, $\bar{\lambda}^i_i = 1$, $(\partial/\partial x + \mathrm{i}k\lambda)u^0_\lambda(x) = 0$, $\psi^0_j = (\delta_{ij}(u^0_{\lambda_j}(x_j)\prod_{l\neq j} u^0_{\lambda^j_l}(x_l)))$. We verify $(L_{\mathrm{I}} + \mathrm{i}K\Lambda)(\psi^0_1, \ldots, \psi^0_n) = 0$.

(ii) We assume for a set $\{\psi_j\}$ of n solutions of (1) a representation of the type $(2b)$:

$$\psi_j = \left[\left(\delta_{ij}u^0_{\lambda_j}(x_j) + \int_{x_j}^{\infty} K^j_i(x_1, \ldots, x_n; y)u^0_{\lambda_j}(y)\right)\prod_{\substack{l\neq j\\ l=1}}^{n} u^0_{\lambda^j_l}(x_l)\right]. \tag{2b}$$

(iii) We assume the boundary conditions

$$\lim_{y\to\infty} u^0_{\lambda_j}(y)K^j_i(x_1, \ldots, x_n; y) = 0,$$

and substituting (2b) into (1) we define $\hat{K}^i_j = K^i_j(y = x_j)$ and obtain

$$\int_{x_j}^{\infty} u^0_{\lambda_j}(y)\left[\left(\frac{\partial}{\partial x_i}+\bar{\lambda}^j_i\frac{\partial}{\partial y}\right)K^j_i - \sum_l q^l_i K^j_l\right] \mathrm{d}y + u^0_{\lambda_j}(x_j)\left[-q^j_i + \bar{\lambda}^j_i \hat{K}^j_i(1-\delta_{ij})\right] \equiv 0.$$

The identity can be satisfied if

$$\left(\frac{\partial}{\partial x_i}+\bar{\lambda}^j_i\frac{\partial}{\partial y}\right)K^j_i = \sum_{m\neq i} \bar{\lambda}^m_i \hat{K}^m_i K^j_m, \qquad q^i_i = 0, \qquad q^i_j = \bar{\lambda}^i_j \hat{K}^i_j. \tag{3b}$$

Let us notice that if $\lambda^i_j \to 0$, then $\bar{\lambda}^i_j = \lambda_j \lambda^{-1}_i$, $u^0_{\lambda^i_j} \to$ constant, in such a way that (2b), (3b) reduce to (2a), (3a), and this formalism is really a generalisation of the previous one recalled in the introduction. Furthermore, if $\bar{\lambda}^i_j = 0$ or $\lambda_j = \lambda^i_j$, then $q^i_j = 0$ in equation (1) without modifying the operator $L_{\mathrm{I}} + \mathrm{i}k\Lambda$.

(iv) Let us consider an integral equation of the equation (4) type written with scalar quantities

$$K^i_j(x_1, \ldots, x_n; y) = \tilde{F}^i_j(x_1, \ldots, x_n; y) + \sum_m \int_{x_m} F^i_m(s; x_1, \ldots, x_n; y) K^m_j(x_1, \ldots, x_n; s),$$

$$\tilde{F}^i_j = F^i_j(s = x_j):$$

$$\left(\frac{\partial}{\partial x_m} + \bar{\lambda}^i_m\frac{\partial}{\partial y} + \bar{\lambda}^j_m\frac{\partial}{\partial s}\right)F^i_j = 0, \qquad m = 1, \ldots, n$$

or (4b)

$$F^i_j = F^i_j\left(x_j - s + \sum_{m\neq j} \bar{\lambda}^j_m x_m;\ x_i - y + \sum_{m\neq i} \bar{\lambda}^i_m x_m\right).$$

If we assume that the solution of (4b) exists and is unique, and further the boundary condition

$$\lim_{s\to\infty} F^i_m(s, x_1, \ldots, x_n; y) K^m_j(x_1, \ldots, x_n; s) = 0,$$

then we can show that the $\{K^i_j\}$ solutions of (4b) satisfy the nonlinear equation (3b). For the proof we notice that

$$\left(\frac{\partial}{\partial x_j} + \bar{\lambda}^i_j\frac{\partial}{\partial y}\right)\tilde{F}^i_j = 0,$$

and applying $\partial/\partial x_j + \bar{\lambda}^i_j\partial/\partial y$ to both sides of the IE (4b) we obtain

$$\left(\frac{\partial}{\partial x_j} + \bar{\lambda}^i_j\frac{\partial}{\partial y}\right)K^i_j = -\tilde{F}^i_j \hat{K}^i_j + \sum_m \left[\int_{x_m} K^m_{j,x_j} F^i_m + \int_{x_m} K^m_j\left(\frac{\partial}{\partial x_j} + \bar{\lambda}^i_j\frac{\partial}{\partial y}\right)F^i_m\right].$$

Taking into account the linear differential equations satisfied by F^i_m following (4b) we find that the RHS can be written

$$\sum_{m\neq j} \bar{\lambda}^m_j \tilde{F}^i_m \hat{K}^m_j + \sum_m \int_{x_j} F^i_m\left(\frac{\partial}{\partial x_j} + \bar{\lambda}^m_j\frac{\partial}{\partial s}\right)K^m_j,$$

and comparing with the solution of the IE (4b), the result (3b) follows.

3.2. Formalism c for n = 2

The study is done in appendix 3. We start with the representation (2c), where we choose $u_j(x_j, x_{j'}) = u^0_{\lambda_j}(x_j)u^0_{\lambda'_j}(x_{j'})$, $j' \neq j$, and obtain our IE which depends upon eight scalar kernels $F^i_{j,p}(i = 1, 2; j = 1, 2; p = 1, 2)$:

$$K^i_{j,p}(x_1, x_2; y) = \tilde{F}^i_{j,p}(x_1, x_2; y) + \sum_{m=1}^{2}\sum_{q=1}^{2}\int_{x_q}^{\infty} F^i_{m,p,q}(s; x_1, x_2; y)K^m_{j,q}(x_1, x_2; s)\,ds,$$

$$\begin{aligned}
&F^i_{j,p}(s_1; s_2; x_1, x_2; y) = F^i_{j,p}(\lambda_j - s_j) + \lambda_{j'}x_{j'} - \lambda^i_{j'}s_{j'}; \gamma^i_{1p}x_1 - y + \gamma^i_{2p}x_2), \qquad j' \neq j,\\
&\tilde{F}^i_{j,p} = F^i_{j,p}(s_1 = x_1, s_2 = x_2), \qquad F^i_{j,p,1} = F^i_{j,p}(s_1 = s, s_2 = x_2),\\
&\qquad F^i_{j,p,2} = F^i_{j,p}(s_1 = x_1, s_2 = s),\\
&\gamma^i_{ii} = 1, \qquad \gamma^i_{ij} = 0, \qquad \gamma^i_{ji} = \bar{\lambda}^i_j, \qquad \gamma^i_{jj} = \lambda_j(\lambda^i_j)^{-1},\\
&\qquad i = 1, 2, \quad j = 1, 2, \quad i \neq j.
\end{aligned} \tag{4c}$$

We remark that from one kernel $F^i_{j,p}$ specifying the restrictions about s_1 and s_2 we build $\tilde{F}^i_{j,p}$ and F^i_{jpq} kernels. The link between the solutions $K^i_{j,p}$ of equation (4c) and the potentials q^i_j of equation (1) is

$$q^i_j = \bar{\lambda}^i_j\hat{K}^i_{j,i} + \hat{K}^i_{j,j}\left(-1 + \frac{\lambda_j}{\lambda^i_j}\right), \qquad \hat{K}^i_{j,p} = K^i_{j,p}(y = x_p), \qquad i \neq j, \quad i = 1, 2. \tag{3c}$$

We remark that the scalar kernels $F^i_{j,p}$ still depend upon two independent variables, and consequently there exist confined solutions in R^2 using for the $F^i_{j,p}$ kernels the same type of confining functions as in the previous paper (Cornille 1978a).

Let us now verify for $n = 2$ that formalism c is really a generalisation of formalism b, or equivalently that (4c) reduces to (4b) when (2c) reduces to (2b). We must have $K^1_{12} = K^1_{22} = K^2_{11} = K^2_{21} = 0$, $F^1_{12} = F^1_{22} = F^2_{11} = F^2_{21} = 0$. All the $F^i_{j,p,q}$ kernels are zero except F^1_{212}, F^1_{111}, F^2_{121}, F^2_{222}. These non-zero kernels are identical to F^1_2, F^1_1, F^2_1, F^2_2 respectively of equation (4b). (We remark that in (4b) we can multiply u^i_j or V^i_j by any constant without altering the partial differential equations that the F^i_j have to satisfy.)

4. Confined solutions in R^n

Starting with formalism b we provide a general method to obtain confined solutions in R^n. We recall that $q^i_j = \bar{\lambda}^i_j\hat{K}^i_j$, so that $q^i_j = 0$ if $\bar{\lambda}^i_j = 0$ or $\lambda^i_j = \lambda_j$. We consider a particular choice of formalism b (we call it bb), in such a way that only a set $q^i_j \not\equiv 0$, whereas all the others are zero:

$q^i_j = 0$, except the set $\{q^{i_1}_{i_2}, q^{i_2}_{i_3}, q^{i_3}_{i_4}, \ldots, q^{i_n}_{i_1}\}$ is $\not\equiv 0$; all $i_1, i_2, \ldots, i_n$ are different.

From the relation between q^i_j and $\hat{K}^i_j$ this means that many $\bar{\lambda}^i_j$ are identically zero, leading to a particular choice of the input representation (2b) which we call (2bb):

$$\begin{aligned}
&\forall_m \neq i_2, \quad \bar{\lambda}^{i_1}_m = 0; \qquad \forall_m \neq i_3, \quad \bar{\lambda}^{i_2}_m = 0; \qquad \forall_m \neq i_4, \quad \bar{\lambda}^{i_3}_m = 0;\\
&\qquad \ldots \forall_m \neq i_1, \quad \bar{\lambda}^{i_n}_m = 0\\
&\bar{\lambda}^{i_1}_{i_2} \neq 0; \qquad \bar{\lambda}^{i_2}_{i_3} \neq 0; \qquad \bar{\lambda}^{i_3}_{i_4} \neq 0; \qquad \bar{\lambda}^{i_n}_{i_1} \neq 0,
\end{aligned} \tag{2bb}$$

We assume that all the $F^i_j = 0$ except those with indices symmetric to the $q^i_j \neq 0$:

$$F^i_j \equiv 0 \text{ except the set } \{F^{i_1}_{i_n}, F^{i_n}_{i_{n-1}}, \ldots, F^{i_4}_{i_3}, F^{i_3}_{i_2}, F^{i_2}_{i_1}\}. \tag{4bb}$$

In fact, in the following, all the results concerning the confinement properties are still valid if we introduce the diagonal kernels $F^j_j \neq 0$. However, for simplicity in the formulae we also put $F^j_j \equiv 0$. Taking (2*bb*) into account in (4*bb*) we obtain

$$\begin{gathered}\{F^{i_1}_{i_n}(x_{i_n} - s + \bar{\lambda}^{i_n}_{i_1} x_{i_1}; x_{i_1} - y + \bar{\lambda}^{i_1}_{i_2} x_{i_2}), F^{i_n}_{i_{n-1}}(x_{i_{n-1}} - s + \bar{\lambda}^{i_{n-1}}_{i_n} x_{i_n}; x_{i_n} - y + \bar{\lambda}^{i_n}_{i_1} x_{i_1}), \ldots, \\ F^{i_3}_{i_2}(x_{i_2} - s + \bar{\lambda}^{i_2}_{i_3} x_{i_3}; x_{i_3} - y + \bar{\lambda}^{i_3}_{i_4} x_{i_4}), F^{i_2}_{i_1}(x_{i_1} - s + \bar{\lambda}^{i_1}_{i_2} x_{i_2}; x_{i_2} - y + \bar{\lambda}^{i_2}_{i_3} x_{i_3})\}.\end{gathered} \tag{4bb}$$

From the expansion (4′) we study first how this confinement property works for the first non-zero contribution to the $q^i_j \neq 0$. Secondly we give explicit examples in closed form corresponding to the sum of all the terms of (4′).

4.1. In order that $\hat{K}^i_j$ be confined in R^n we must at least have that this property holds for the first contributing term in the expansion (4′)

In order to have some insight let us start with a simple example, given by (7) ($F^1_2 \neq 0$, $F^2_3 \neq 0$, $F^3_1 \neq 0$), and look at K^2_1, K^1_3, K^3_2 corresponding to the $q^i_j \neq 0$:

$$\begin{pmatrix} *, & *, & K^1_3 \\ K^2_1, & *, & * \\ *, & K^3_2, & * \end{pmatrix} = \begin{pmatrix} * & * & \int F^1_2 \tilde{F}_3 \\ \int F^2_3 \tilde{F}^3_1, & *, & * \\ * & \int F^3_1 \tilde{F}^1_2, & * \end{pmatrix} + \text{higher-order terms.}$$

We remark that the smallest-order contribution to these K^i_j appears first in $\int \mathcal{F}\tilde{\mathcal{F}}$. If further the F^i_j are degenerate, i.e. a product of a function of s and a function of y, then $\int F^i_k \tilde{F}^k_j$ is in fact a product of three different functions. If the variables associated to these functions represent a basis in R^3, and if each function vanishes when its corresponding variable goes to $\pm\infty$, then we can hope to get terms $\int F^i_k \tilde{F}^k_j$ confined in R^3. If now the F^i_j are a sum of such degenerate terms, $F^i_j = \Sigma_m\, g^i_{j,m}(u^i_j(s)) h^i_{j,m}(v^i_j(y))$, then $\int F^i_k \tilde{F}^k_j$ becomes a sum of terms, each term being as above a product of three different functions associated with different variables. In this case also we hope to obtain $\int F^i_k \tilde{F}^k_j$ confined in R^3.

Coming back now to the general formalism *bb*, we remark that the K^i_j with indices $(i, j) = (i_1, i_2) \ldots (i_n, i_1)$ symmetric to the $F^i_j \neq 0$, $(i, j) = (i_2, i_1) \ldots (i_1, i_n)$, appear firstly in $\mathcal{K}_{n-1}$, the $(n-1)$th order of perturbation (4′). Let us write $F^i_j = F^i_j(s; y)$ (forgetting for the moment the x_i), to obtain

$$\begin{aligned}\hat{K}^{i_1}_{i_2} = \int \ldots \int \prod_{l=3}^{n} \mathrm{d}s_l \theta(s_l - x_{i_l}) F^{i_1}_{i_n}(s_n, y) F^{i_n}_{i_{n-1}}(s_{n-1}, s_n) \ldots F^{i_4}_{i_3}(s_3, s_4) \\ \times F^{i_3}_{i_2}(s = x_{i_2}, s_3) + \text{higher-order terms.}\end{aligned}$$

Let us now in (4*bb*) consider the degenerate kernels $F^i_j = g^i_j h^i_j$ and define $\Lambda^i_{j,k}$:

$$F^{i_2}_{i_1} = g^{i_2}_{i_1}(x_{i_1} - s + \bar{\lambda}^{i_1}_{i_2} x_{i_2}) h^{i_2}_{i_1}(-y + \bar{\lambda}^{i_2}_{i_3} x_{i_3}), \qquad i_1 \to i_2, \quad i_2 \to i_3, \quad i_3 \to i_4,$$

then $F^{i_2}_{i_1} \to F^{i_3}_{i_2}$ and so on;

$$A^i_{j,k}(\bar{\lambda}^k_i x_i) = \int_{-\bar{\lambda}^k_i x_i}^{\infty} g^i_k(-w) h^k_j(-w)\, \mathrm{d}w.$$

Then $\hat{K}^{i_1}_{i_2}$ is given by

$$\begin{aligned} \hat{K}^{i_1}_{i_2} &= \hat{N}^{i_1}_{i_2} + \text{other terms}, \\ \hat{N}^{i_1}_{i_2} &= A^{i_1}_{i_{n-1},i_n}(\lambda^{i_n}_{i_1} x_{i_1}) h^{i_1}_{i_n}(\lambda^{i_1}_{i_2} x_{i_2}) g^{i_3}_{i_2}(\lambda^{i_2}_{i_3} x_{i_3}) \prod_{j=4}^{n} A^{i_j}_{i_{j-2},i_{j-1}}(\lambda^{i_{j-1}}_{i_j} x_{i_j}). \end{aligned} \tag{10}$$

Now let us assume, as in equation (8),

$$g^i_j(u) \underset{|u|\to\infty}{\to} 0,\ h^i_j(u) \underset{|u|\to\infty}{\to} 0,\ A^{i_j}_{i_{j-2},i_{j-1}}(u) \underset{|u|\to\infty}{\to} 0 \qquad \text{or} \qquad \int_{-\infty}^{+\infty} g^{i_j}_{i_{j-1}}(u) h^{i_{j-1}}_{i_{j-2}}(u) = 0. \tag{8}$$

Then $\hat{N}^{i_1}_{i_2}$ is the product of n different functions, each of them depending upon one variable and going to zero when this variable goes to $\pm\infty$. Consequently $\hat{N}^{i_1}_{i_2}$ is confined in R^n.

4.2. Simple examples for the formalism bb where the solution can be written in closed form

Now what happens for the other-order terms of expansion (4′). For the most simple degenerate kernels $F^i_j = g^i_j(u^i_j(s)) h^i_j(v^i_j(y))$, the sum of these terms (solution of equation (4)) can be written in closed form, and we find

$$q^{i_1}_{i_2} = \lambda^{i_1}_{i_2} \hat{K}^{i_1}_{i_2}, \qquad \hat{K}^{i_1}_{i_2} = \hat{N}^{i_1 q}_{i_2} D^{-1}, \qquad D = 1 - \prod_{j=1}^{n} A^{i_j}_{i_{j-2},i_{j-1}}(\lambda^{i_{j-1}}_{i_j} x_{i_j}),$$

where $\hat{N}^{i_1}_{i_2}$ is given by equation (10). Consequently (putting aside the possible zeros of the Fredholm determinant D), D is bounded and $\hat{K}^{i_1}_{i_2}$ is confined in R^n if we assume that the F^i_j kernels are of the type given by equation (8). Let us remark that in order to satisfy equation (8) for $A^i_{j,k}$ it is sufficient that $g^i_k(w) h^k_j(w)$ be an integrable antisymmetric function of w.

Is this confinement property restricted to the most simple degenerate kernels? We shall see that this property is more general and for simplicity come back to the $n = 3$ case with the example where $\mathscr{F}$ is given by (7). We assume

$$F^i_j = \sum_{m=1}^{m_0} g^i_{j,m}(x_j - s + \lambda^j_i x_i) h^i_{j,m}(x_i - y + \lambda^i_k x_k), \qquad (i, j) = (1, 2), (2, 3), (3, 1),$$

whereas all the other F^i_j kernels are zero. The solution for $m_0 = 1$ was written (7*bb*). For $m_0 > 1$ we get for the non-nulls q^i_j, $(j, i) = (2, 1), (1, 3), (3, 2)$

$$\hat{K}^j_i = \sum_{m,p} h^j_{k,m}(\lambda^j_i x_i) A^{j,m,p}_{i,k}(\lambda^k_j x_j)\Big(g^k_{i,p}(\lambda^i_k x_k) + \sum_q A^{k,p,q}_{j,i}(\lambda^i_k x_k) B_q\Big),$$

$$\begin{aligned} B_q\Big(1 - \sum_{m,p} A^{i,q,m}_{k,j}(\lambda^j_i x_i) A^{j,m,p}_{i,k}(\lambda^k_j x_j) A^{k,p,q}_{j,i}(\lambda^i_k x_k)\Big) \\ - \sum_{q' \neq q} B_{q'} \sum_{m,p} A^{i,q,m}_{k,j}(\lambda^j_i x_i) A^{j,m,p,}_{i,k}(\lambda^k_j x_j) A^{kpq'}_{j,i}(\lambda^i_k x_k) \\ = \sum_{m,p} g^k_{i,p}(\lambda^i_k x_k) A^{i,q,m}_{k,j}(\lambda^j_i x_i) A^{j,m,p}_{i,k}(\lambda^k_j x_j), \qquad i, j, k \text{ all different} \end{aligned}$$

where

$$A_{i,k}^{j,p,q}(x)=\int_{-x}^{\infty} g_{k,p}^{i}(-u)h_{j,q}^{k}(-u)\,\mathrm{d}u.$$

Let us still assume equation (8):

$$g_{j,m}^{i}(u)\underset{|u|\to\infty}{\to}0, \qquad h_{j,m}^{i}(u)\underset{|u|\to\infty}{\to}0, \qquad A_{jk}^{ipq} \text{ is bounded,}$$

$$A_{i,k}^{j,p,q}(x)\underset{|x|\to\infty}{\to}0, \qquad \text{or} \qquad \int_{-\infty}^{+\infty} g_{k,p}^{i}(u)h_{j,p}^{k}(u)\,\mathrm{d}u=0.$$

In the first relation (assuming that B_q is bounded in R^3) we see that $\hat{K}_i^j$ is a sum of products of functions depending respectively on x_i, x_j, x_k, each of them going to zero when respectively $|x_i|\to\infty$, $|x_j|\to\infty$, $|x_k|\to\infty$. The second relation is an algebraic linear system with coefficients bounded in R^3. The solutions of this system give the B_q, and if we exclude the zeros of the determinant (which is in fact the Fredholm determinant of equation (4), (4*bb*)), then the B_q are bounded in R^3. In conclusion the $q_j^i\neq 0$ are confined in R^3. In order to satisfy equation (8), we can for instance take very simple examples,

$$g_{i,m}^{i}(u)=u^{\alpha_{j,m}^{i}}\exp(-u^{\beta_{j,m}^{i}}), \qquad h_{j,m}^{i}(u)=u^{\nu_{j,m}^{i}}\exp(-u^{\eta_{j,m}^{i}}),$$

where $\alpha_{j,m}^{i}$, $\beta_{j,m}^{i}$, $\nu_{j,m}^{i}$, $\eta_{j,m}^{i}$ are positive integers, $\beta_{j,m}^{i}$ and $\eta_{j,m}^{i}$ being even integers. In order that $g_{j,m}^{i}(u)h_{k,p}^{j}(u)$ be an antisymmetric function of u, it is sufficient to consider $\alpha_{j,m}^{i}$ odd (even) integers and $\nu_{j,p}^{i}$ even (odd) integers.

5. The IE resulting from a compatibility condition between different linear differential systems

In this section we show that our IEs of the equation (4) type for $n\geqslant 3$ are common to different linear differential systems, in such a way that the reconstructed potentials from these IEs must satisfy the nonlinear constraints coming from the compatibility conditions between these systems.

5.1. *Formalism a and n = 3*

Let us assume that both system (1) and another system (1′) *have the same solution* ψ: $(L_{\mathrm{I}}+\mathrm{i}k\Lambda-Q_{\mathrm{I}})\psi=0(1)$, $(L_{\mathrm{II}}-Q_{\mathrm{II}})\psi=0(1')$, where

$$L_{\mathrm{I}}=\left(\delta_{ij}\frac{\partial}{\partial x_j}\right), \qquad L_{\mathrm{II}}=\left(\delta_{ij}\left(\frac{\lambda_j}{\lambda_k}\frac{\partial}{\partial x_k}-\frac{\lambda_j}{\lambda_l}\frac{\partial}{\partial x_l}\right)\right),$$

with $k=j+1$, $l=j+2$, $l=1$ if $j=2$ and $l=2$, $k=1$ if $j=3$,

$$\Lambda=\begin{pmatrix}\lambda_1 & & 0\\ & \lambda_2 & \\ 0 & & \lambda_3\end{pmatrix}, \qquad Q_{\mathrm{I}}=\begin{pmatrix}0 & q_1^2 & q_1^3\\ q_2^1 & 0 & q_2^3\\ q_3^1 & q_3^2 & 0\end{pmatrix}, \qquad Q_{\mathrm{II}}=\begin{pmatrix}0 & -q_1^2 & q_1^3\\ q_2^1 & 0 & -q_2^3\\ -q_3^1 & q_3^2 & 0\end{pmatrix}.$$

As we shall see, the IE (4*a*) is common to (1) and (1′), and the nonlinear constraints (5*a*) which are required for the q_j^i reconstructed from IE (4*a*) represent the compatibility condition between (1) and (1′).

(i) *Compatibility between the two systems.* Let us define $L^{\pm}=L_{\rm I}\pm L_{\rm II}$, $2Q^{\pm}=Q_{\rm I}\bullet Q_{\rm II}$, $\Lambda_{\pm}$:

$$Q^{+}=\begin{pmatrix}0&0&q_1^3\\ q_2^1&0&0\\ 0&q_3^2&0\end{pmatrix},\qquad Q^{-}=\begin{pmatrix}0&q_1^2&0\\ 0&0&q_2^3\\ q_3^1&0&0\end{pmatrix},$$

$$\Delta_{+}=\begin{pmatrix}\lambda_3&&0\\ &\lambda_1&\\ 0&&\lambda_2\end{pmatrix},\qquad \Lambda_{-}=\begin{pmatrix}\lambda_2&&0\\ &\lambda_3&\\ 0&&\lambda_1\end{pmatrix}.$$

The compatibility condition is $[L^{+},L^{-}]\psi=0$. Taking into account the relations $L^{\pm}(Q^{\mp}\psi)=(L^{\pm}Q^{\mp})\psi+Q^{\mp}\Lambda_{\pm}\Lambda^{-1}(L^{\mp}\psi)$, we can eliminate ψ to obtain

$$-\mathrm{i}k\{\Lambda Q^{-}-\Lambda Q^{+}+Q^{+}\Lambda_{-}-Q^{-}\Lambda_{+}\}$$
$$+\{L^{-}Q^{+}-L^{+}Q^{-}+2Q^{+}\Lambda_{-}\Lambda^{-1}Q^{+}-2Q^{-}\Lambda_{+}\Lambda^{-1}Q^{-}\}\equiv 0.$$

It is easy to verify that the first bracket is identically zero. The second bracket written with scalar quantities leads for the $\{q_j^i\}$ to the nonlinear three-wave equation ($5a$).

(ii) IE *associated to system* (1′). We assume that (1′) has a set of solutions ψ_j given by ($2a$). We substitute ($2a$) into system (1′) following the general method explained in § 2.4 and find that the transforms K_j^i must satisfy (for simplicity we do not reproduce the boundary conditions)

$$\int_{x_l} u^0_{\lambda_l}(y)\left[\left(\frac{\lambda_j}{\lambda_k}\frac{\partial}{\partial x_k}-\frac{\lambda_l}{\lambda_l}\frac{\partial}{\partial x_l}\right)K_i^j+K_k^iq_i^k-K_l^iq_i^l\right]\mathrm{d}y\equiv 0,\qquad k\neq j,\quad l\neq i,$$

$$u^0_{\lambda_i}(x_i)\left(q_i^j(1-\delta_{ij})-\hat K_i^j\frac{\lambda_i}{\lambda_j}\right)=0,\qquad i\neq j.$$

Thus we still find for the potentials $q_i^j=\hat K_i^j\lambda_i/\lambda_j$, and the IE associated to (1′) if it exists is such that the solutions K_j^i must satisfy

$$\left(\frac{1}{\lambda_k}\frac{\partial}{\partial x_k}-\frac{1}{\lambda_l}\frac{\partial}{\partial x_l}\right)K_j^i+\frac{1}{\lambda_k}K_k^i\hat K_j^k-\frac{1}{\lambda_l}K_l^i\hat K_j^l=0. \tag{11a}$$

(iii) *Nonlinear constraints satisfied by the solutions of the* IE ($4a$) *associated to system* (1). We report here briefly results obtained in the previous paper (Cornille 1978a). Owing to the fact that the operator $(\lambda_k^{-1}\partial/\partial x_k-\lambda_l^{-1}\partial/\partial x_l)$, $k\neq j$, $l\neq j$, when applied to F_j^i or $\tilde F_j^i$ gives zero, we deduce from equation ($4a$) that

$$\left(\lambda_k^{-1}\frac{\partial}{\partial x_k}-\lambda_l^{-1}\frac{\partial}{\partial x_l}\right)K_j^i$$
$$=\lambda_l^{-1}\tilde F_l^i\hat K_j^l-\lambda_k^{-1}\tilde F_k^i\hat K_j^k+\sum_m\int_{x_m}F_m^i\left(\lambda_k^{-1}\frac{\partial}{\partial x_k}-\lambda_l^{-1}\frac{\partial}{\partial x_l}\right)K_j^m.$$

If we compare this with the solution of ($4a$) itself we see that the K_j^i satisfy equation ($11a$). Consequently equation ($4a$) is also an IE for system (1′). If, further, between equation ($3a$) and ($11a$) we eliminate the K_i^j term, then we obtain the nonlinear three-wave equation written down in ($5a$). In conclusion, (1) and (1′) have the same IE of the equation ($4a$) type, and the reconstructed potentials must satisfy the nonlinear equation ($5a$) which results from the compatibility between both systems.

5.2. *Formalism a and $n=4$*

Let us assume that ψ is a common solution to three systems: $(L_{\mathrm{I}}+\mathrm{i}k\Lambda-Q_{\mathrm{I}})\psi=0(1)$, $(L_{\mathrm{II}}-Q_{\mathrm{II}})\psi=0(1')$, $(L_{\mathrm{III}}-Q_{\mathrm{III}})\psi=0(1'')$, where

$$L_{\mathrm{I}}\left(\delta_{ij}\frac{\partial}{\partial x_i}\right),\quad L_{\mathrm{II}}=\left(\delta_{ij}\left(\frac{\lambda_j}{\lambda_k}\frac{\partial}{\partial x_k}-\frac{\lambda_j}{\lambda_l}\frac{\partial}{\partial x_l}\right)\right),$$

with $k=j+1$, $l=j+2$, $l=1$ if $j=3$ and $l=2$, $k=1$ if $j=4$,

$$L_{\mathrm{III}}=\left(\delta_{ij}\left(\frac{\lambda_j}{\lambda_k}\frac{\partial}{\partial x_k}-\frac{\lambda_j}{\lambda_l}\frac{\partial}{\partial x_l}\right)\right),$$

with $k=j+2$, $l=j+3$, $l=1$ if $j=2$, $l=2$, $k=1$ if $j=3$ and $l=3$, $k=2$ if $j=4$,

$$\Lambda=\begin{pmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \lambda_4\end{pmatrix},\qquad Q_{\mathrm{I}}=\begin{pmatrix}0 & q_1^2 & q_1^3 & q_1^4\\ q_2^1 & 0 & q_2^3 & q_2^4\\ q_3^1 & q_3^2 & 0 & q_3^4\\ q_4^1 & q_4^2 & q_4^3 & 0\end{pmatrix},$$

$$Q_{\mathrm{II}}=\begin{pmatrix}0 & -q_1^2 & q_1^3 & 0\\ 0 & 0 & -q_2^3 & q_2^4\\ q_3^1 & 0 & 0 & -q_3^4\\ -q_4^1 & q_4^2 & 0 & 0\end{pmatrix},\qquad Q_{\mathrm{III}}=\begin{pmatrix}0 & 0 & -q_1^3 & q_1^4\\ q_1^2 & 0 & 0 & -q_2^4\\ -q_3^1 & q_3^2 & 0 & 0\\ 0 & -q_4^2 & q_4^3 & 0\end{pmatrix}.$$

As we shall see, the IE ($4a$) is common to (1), (1') and (1''), and the nonlinear constraints ($6a$) for the q_j^i represent the compatibility condition between the three systems.

(i) *Compatibility conditions between the three systems.* Let us define $L^{\pm}=L_{\mathrm{I}}\pm L_{\mathrm{II}}$, $M^{\pm}=L_{\mathrm{I}}\pm L_{\mathrm{III}}$. We have two compatibility conditions, $[L^+,L^-]\psi=0$ and $[M^+,M^-]\psi=0$ which are studied in appendix 1. It is shown that the scalar kernels q_i^i satisfy the nonlinear equations ($6a$). Consequently the IE ($4a$) must be a common IE to the three systems (1), (1') and (1'')

(ii) IE *associated to the systems* (1') *and* (1''). We sketch very briefly the results using the general method (§ 2.2). On the one hand we assume for both (1') and (1'') that the ψ_j given by ($2a$) are solutions. We find that the transforms K_j^i must satisfy well-defined nonlinear equations:

$$0=\left(\lambda_k^{-1}\frac{\partial}{\partial x_k}-\lambda_q^{-1}\frac{\partial}{\partial x_q}\right)K_j^i+\frac{1}{\lambda_k}K_k^i\hat{K}_j^k-\frac{1}{\lambda_q}K_q^i\hat{K}_j^q,\qquad k\neq j,\quad q\neq j,\quad q_j^i=\lambda_j\lambda_i^{-1}\hat{K}_j^i.$$

On the other hand we consider the IE ($4a$). From the fact that the operator $(\lambda_j^{-1}\partial/\partial x_k-\lambda_q^{-1}\partial/\partial x_q)$ gives zero when applied to $\tilde{F}_j^i$ or F_j^i, we deduce that the solutions of the IE ($4a$) satisfy the above nonlinear equations. In conclusion, ($4a$) is an IE common to (1), (1') and (1'').

5.3. Formalism bb and $n=3$

Let us assume that both system (1) and another system (1′) have the same solution ψ: $(L_{\rm I}+ik\Lambda_{\rm I}-Q_{\rm I})\psi=0(1)$, $(L_{\rm II}+ik\Lambda_{\rm II}-Q_{\rm II})\psi=0(1')$, where

$$L_{\rm I}=\left(\delta_{ij}\frac{\partial}{\partial x_i}\right),\qquad L_{\rm II}=\begin{pmatrix}\partial/\partial x_2 & & 0\\ & \partial/\partial x_3 & \\ 0 & & \partial/\partial x_1\end{pmatrix},\qquad Q_{\rm I}=\begin{pmatrix}0 & q_1^2 & 0\\ 0 & 0 & q_2^3\\ q_3^1 & 0 & 0\end{pmatrix},$$

$$Q_{\rm II}=\begin{pmatrix}0 & 0 & q_1^3\\ q_2^1 & 0 & 0\\ 0 & q_3^2 & 0\end{pmatrix},\qquad \Lambda_{\rm I}=\begin{pmatrix}\lambda_1 & & 0\\ & \lambda_2 & \\ 0 & & \lambda_3\end{pmatrix},\qquad \Lambda_{\rm II}=\begin{pmatrix}\lambda_2 & & 0\\ & \lambda_3 & \\ 0 & & \lambda_1\end{pmatrix}.$$

(i) *Compatibility between the two systems.* We consider $[L_{\rm I},L_{\rm II}]\psi=0$, and using the identities $L_{\rm II}(Q_{\rm I}\psi)-(L_{\rm II}Q_{\rm I})\psi-Q_{\rm I}(L_{\rm I}\psi)=0$, $L_{\rm I}(Q_{\rm II}\psi)-(L_{\rm I}Q_{\rm II})\psi-Q_{\rm II}(L_{\rm II}\psi)=0$, the compatibility condition is written as a sum of two terms. The first term $-ik[\Lambda_{\rm I}Q_{\rm II}+Q_{\rm I}\Lambda_{\rm I}-\Lambda_{\rm II}Q_{\rm I}-Q_{\rm II}\Lambda_{\rm II}]$ is identically zero. The second term $[L_{\rm II}Q_{\rm I}-L_{\rm I}Q_{\rm II}+(Q_{\rm I})^2-(Q_{\rm II})^2]=0$ give the nonlinear constraints for the potentials $(\partial/\partial x_i)q_j^i=q_k^iq_j^k$. Let us remark that if $q_j^i\in Q_{\rm I}$ (or $Q_{\rm II}$), then q_k^i and q_j^k belong to $Q_{\rm II}$ (or $Q_{\rm I}$). Thus the compatibility constraints mix in the relation $q_{j,x_i}^i=q_k^iq_j^k$, the potentials of systems (1) and (1′). If we want to get constraints for potentials $\in Q_{\rm I}$ (or $Q_{\rm II}$), we must eliminate the other potentials. In this way instead of having nonlinear three-wave type equations we have integro-differential equations. For instance, for $q_j^i\in Q_{\rm I}$, where we know that they can be confined in R^3, we can integrate to obtain

$$q_{j,x_i}^i=\int_{x_k}^{\infty}q_j^iq_k^j\,{\rm d}x_k'\int_{x_j}^{\infty}q_j^iq_i^k\,{\rm d}x_j'\ (5bb),$$

written down previously in § 2.3.

Let us recall that, in order to reconstruct the potentials of (1), the kernels F_j^i of the IE (4*bb*) are of the type

$$F_j^i(x_i-s+\bar\lambda_i^jx_i;\,x_i-y+\bar\lambda_k^ix_k),\qquad (i,j)=(1,2),(2,3),(3,1),\qquad (4bb)$$

whereas $\bar\lambda_2^1=\bar\lambda_3^2=\bar\lambda_1^3=0$ and $q_i^j=\bar\lambda_i^j\hat K_i^j$ for the same set of (i,j). As we shall see in the following, this IE with the same kernels F_j^i is also an IE associated to (1′)

(ii) IE *associated to* (1′). Firstly we assume that (1′) has a set of solutions ψ_j given by (2*bb*) with $\bar\lambda_2^1=\bar\lambda_3^2=\bar\lambda_1^3=0$. We substitute (2*bb*) into (1) to obtain the set of nonlinear equations that the transforms K_j^i must satisfy,

$$\int_{x_j}u_j\left[\left(\frac{\partial}{\partial x_k}+\bar\lambda_k^i\frac{\partial}{\partial y}\right)K_i^j-K_l^jq_i^l\right){\rm d}y+u_j(-q_i^j+\hat K_i^l\bar\lambda_k^l)\delta_{j+1,i}\equiv 0,$$

where $k=i+1$, $l=i+2$ if $i=2$, $l=1$ if $i=3$, and $k=1$, $l=2$, $\delta_{j+1,i}=\delta_{1,i}$ if $j=3$. Consequently the K_j^i must satisfy

$$\left(\frac{\partial}{\partial x_k}+\bar\lambda_k^i\frac{\partial}{\partial y}\right)K_j^i=K_l^i\hat K_j^l\bar\lambda_k^l,\qquad k=j+1,\quad l=j+2,\qquad (11bb)$$

where $k=1$, $l=2$ if $j=3$ and $l=1$ if $j=2$. Moreover, $q_2^1=\bar\lambda_3^1\hat K_2^1$, $q_3^2=\bar\lambda_1^2\hat K_3^2$, $q_1^3=\bar\lambda_2^3\hat K_1^3$ and $\bar\lambda_j^l=1$.

Secondly we want to show that the solutions K_j^i of the IE (4*bb*) also satisfy the nonlinear equation (11*bb*). For this we apply the operator $(\partial/\partial x_k + \lambda_k^i \partial/\partial y)$ to both sides of the IE (4) with (4*bb*) kernels to obtain

$$\left(\frac{\partial}{\partial x_k} + \lambda_k^i \frac{\partial}{\partial y}\right) K_j^i = \tilde{F}_l^i K_j^l \lambda_k^l + \sum \int F_m^i \left(\frac{\partial}{\partial x_k} + \lambda_k^m\right) K_j^m,$$

where k and l have been defined in (11*bb*). Comparing with the solutions of the IE (4*bb*) we see that these solutions satisfy (11*bb*). In conclusion, the IE (4) with F_j^i kernels (4*bb*) is associated to both systems (1) and (1′), and the K_j^i solutions must satisfy both equation (11*bb*) and equation (3*bb*), which we rewrite as

$$\left(\frac{\partial}{\partial x_j} + \lambda_j^i \frac{\partial}{\partial y}\right) K_j^i = K_k^i \lambda_j^k \hat{K}_j^k, \qquad k = j+1, \quad k = 1 \quad \text{if} \quad j = 3, \tag{3bb}$$

whereas the potentials of Q_{I} are $q_1^2 = \lambda_1^2 \hat{K}_1^2$, $q_3^1 = \lambda_3^1 \hat{K}_3^1$ and $q_2^3 = \lambda_2^3 \hat{K}_2^3$. Combining (3*bb*) and (11*bb*) we obtain for the six potentials of Q_{I} and Q_{II} the relations $(\partial/\partial x_i) q_j^i = q_k^i q_j^k$ which were obtained previously from the compatibility conditions between the two systems (1) and (1′). Let us finally remark that the solutions (7*bb*) confined in R^3 correspond to the three potentials of Q_{I}, and consequently the nonlinear integro-differential equations (5*bb*) have solutions confined in R^3.

5.4. *Formalism b and n = 3*

We sketch the results very briefly, because nothing really new appears in this case. As for formalism *a* we still obtain a three-wave nonlinear equation resulting from the constraints equations of the solutions of the IE (4*b*) and from the compatibility conditions between two linear systems. The study is done in appendix 2, and we obtain

$$\left(\frac{1}{\lambda_j^i} \frac{\partial}{\partial x_j} + \frac{\lambda_k^j}{\lambda_k^i \lambda_i^j} \frac{\partial}{\partial x_i} - \frac{1}{\lambda_k^i} \frac{\partial}{\partial x_k}\right) \hat{K}_j^i = \hat{K}_k^i \hat{K}_j^k \left(\frac{\lambda_j^k}{\lambda_j^i} + \frac{\lambda_k^i \lambda_i^k}{\lambda_k^i \lambda_i^j}\right). \tag{5b}$$

It is easy to verify that, when $\lambda_j^i \to 0$, (5*b*) reduces to (5*a*). Here, as in the previous paper (Cornille 1978a), we could use formalism *b* to show that if we interpret one co-ordinate as a finite time then there exist solutions of equation (5*b*) which are confined in the R^2 space spanned with the two other coordinates.

6. Conclusions

A clear understanding of the properties of multidimensional nonlinear evolution equations (for instance the confinement property) requires as a preliminary study the investigation of the multidimensional space in which the solutions have to evolve. The nonlinear equations called 'integrable' or 'linearisable' or 'soluble by the inverse method' are in fact those which represent a compatibility condition between different linear partial differential systems (Ablowitz and Haberman 1975). In this case the main problem is to find a formalism (termed IE) generating potentials associated with linear systems. In the R^1 space case (for simple systems), the classical method is the analytic one. Unfortunately in the R^n space case this method is not yet available (it is not even so clear that we have really to do it). In the previous paper as well as in the present paper we have considered an algebraic method (Zakharov and Shabat (1974) were the first to realise the usefulness of an algebraic approach) which generates a class of potentials associated to a linear differential system and which can be studied in R^n. In this way we have found specific features of the R^n case not present in R^1:

(i) The non-uniqueness of the IE appears at two stages. Firstly, it results from the choice of the set of solutions that we want to reconstruct. This is illustrated here in what we call formalisms a and b. One generalises the other and has new degrees of freedom: we can for instance introduce arbitrary parameters or put to zero some of the scalar potentials. Let us remark that all the corresponding IEs reduce to only one equation in R—the classical Marchenko equation. We have even written down here (formalism c) in the 2×2 case a general IE with eight arbitrary scalar kernels which, for instance, could generate a more general class of solutions of the generalised nonlinear Schrödinger equation than the one deduced previously (Cornille 1978b). The second ambiguity comes from the fact that we do not know if the IE determined is the more general one associated to a given input representation of the solutions of the system. In some cases we know (Cornille 1978a) that there exist at least two IEs corresponding to the same input representation, one generalising the other. Here, the fact that for $n \geqslant 3$ the reconstructed potentials satisfy constraints suggests that there can exist a generalisation where these constraints disappear.

(ii) The confinement property in R comes from the fact that the degenerate kernels of the IE are of the exponential type, whereas in R^n the extension of this property with the same type of kernels does not work. However, using a limiting process, Manakov *et al* (1977) have shown, for the exponential-type kernels, how to obtain confined solutions, but this possibility is not specific to the exponential kernels (Cornille 1979). In R^2 the confinement property is realised naturally by the fact that the IEs include a large class of other kernels, such as Gaussian kernels.

In R^n, $n>2$ we give a general procedure for constructing potentials confined in all asymptotic directions of space. However, the potentials are constraints, such that these confined solutions correspond to a new class of nonlinear integro-differential equations solvable by the inverse method. Some of the scalar potentials must be zero, so that we take advantage of the degrees of freedom of our generalised IE (formalism b).

(iii) For $n \geqslant 3$ the IEs that we have constructed as being associated to a particular linear system are in fact the IEs common to other linear systems. This new aspect of the IE was not present in one dimension or in the two-dimensional case presented here. For $n=3$ the two systems are first-order partial differential systems, and from the work of Ablowitz and Haberman (1975) it is clear that the compatibility condition must lead to the two-spatial-dimensional nonlinear three-wave equation (a physical equation of plasma physics). Thus from the formalism of the previous paper or formalism b of the present paper we know that this nonlinear equation has an infinite number of confined solutions in R^2 for any finite time. For formalism bb the potentials associated with the two linear systems belong to two distinct classes, in such a way that the resulting compatibility condition leads to some kind of nonlinear three-wave equation mixing the two classes. If we eliminate one class, then the resulting equation becomes an integro-differential one of a new type which has confined solutions in R^3. For $n=4$ the IE is common to three systems, and the constraints that the potentials must satisfy represent their compatibility conditions, and so on for higher values of n.

In conclusion, exploring, for the IE, the reconstructed potential space associated to linear differential systems in R^n we obtain results unexpected from our knowledge of the one-dimensional case. Moreover, new features appear from $n=2$ to $n=3$.

Appendix 1

We study the compatibility condition in the $n=4$ case.

(i) We consider the two systems $L^{\pm}\psi=(-ik\Lambda+2Q^{\pm}+R^{+})\psi$, $L^{\pm}=L_{\mathrm{I}}\pm L_{\mathrm{II}}$, where

$$R^{-}=Q^{+}=\begin{pmatrix}0&0&q_1^3&0\\0&0&0&q_2^4\\q_3^1&0&0&0\\0&q_4^2&0&0\end{pmatrix},\qquad Q^{-}=\begin{pmatrix}0&q_1^2&0&0\\0&0&q_2^3&0\\0&0&0&q_3^4\\q_4^1&0&0&0\end{pmatrix},$$

$$R^{+}=\begin{pmatrix}0&0&0&q_1^4\\q_2^1&0&0&0\\0&q_3^2&0&0\\0&0&q_4^3&0\end{pmatrix},\qquad L_{\mathrm{I}}=(\delta_{ij}\partial/\partial x_i),\qquad L_{\mathrm{II}}=(\delta_{ij}\partial_i),\tag{A1.1}$$

$$\partial_1=\frac{\lambda_1}{\lambda_2}\frac{\partial}{\partial x_2}-\frac{\lambda_1}{\lambda_3}\frac{\partial}{\partial x_3},\qquad \partial_2=\frac{\lambda_2}{\lambda_3}\frac{\partial}{\partial x_3}-\frac{\lambda_2}{\lambda_4}\frac{\partial}{\partial x_4},$$

$$\partial_3=\frac{\lambda_3}{\lambda_4}\frac{\partial}{\partial x_4}-\frac{\lambda_3}{\lambda_1}\frac{\partial}{\partial x_1},\qquad \partial_4=\frac{\lambda_4}{\lambda_1}\frac{\partial}{\partial x_1}-\frac{\lambda_4}{\lambda_2}\frac{\partial}{\partial x_2},$$

ψ being a column vector, and study the compatibility condition $[L^{+},L^{-}]\psi=0$, which gives

$$0=-ik\Lambda(Q^{-}-Q^{+})+L^{-}(Q^{+}\psi)-L^{+}(Q^{-}\psi)+\tfrac{1}{2}[L^{-}(R^{+}\psi)-L^{+}(R^{+}\psi)].\tag{A1.2}$$

Let us introduce the matrices Λ_i associated to Λ:

$$\Lambda=\begin{pmatrix}\lambda_1&&&0\\&\lambda_2&&\\&&\lambda_3&\\0&&&\lambda_4\end{pmatrix},\qquad \Lambda_1=\begin{pmatrix}\lambda_2&&&0\\&\lambda_3&&\\&&\lambda_4&\\0&&&\lambda_1\end{pmatrix},$$

$$\Lambda_2=\begin{pmatrix}\lambda_3&&&0\\&\lambda_4&&\\&&\lambda_1&\\0&&&\lambda_2\end{pmatrix},\qquad \Lambda_3=\begin{pmatrix}\lambda_4&&&0\\&\lambda_3&&\\&&\lambda_2&\\0&&&\lambda_1\end{pmatrix}.$$

With some algebra we get

$$\tfrac{1}{2}(L^{-}(R^{+}\psi)-L^{+}(R^{+}\psi))=-((L_{\mathrm{II}}R^{+})+R^{+}\Lambda_1\Lambda^{-1}(R^{+}-Q^{+}))\psi,$$

$$L^{-}(Q^{+}\psi)=[L^{-}Q^{+}+Q^{+}\Lambda_2\Lambda^{-1}(-ik\Lambda+2R^{+}+Q^{-})]\psi,$$

$$L^{+}(Q^{-}\psi)=[L^{+}Q^{-}+Q^{-}\Lambda_3\Lambda^{-1}(-ik\Lambda+2Q^{-}+Q^{+})]\psi.$$

Taking into account these relations, the RHS of (A1.2) becomes the sum of two terms. The first, $-ik[\Lambda Q^{-}-\Lambda Q^{+}+Q^{+}\Lambda_2-Q^{-}\Lambda_3]$, is identically zero, so that the compatibility condition between the two systems (A1.1) is finally

$$L^{-}Q^{+}-L^{+}Q^{-}-L_{\mathrm{II}}R^{+}+Q^{+}\Lambda_2\Lambda^{-1}(Q^{-}+2R^{+})$$
$$-Q^{-}\Lambda_3\Lambda^{-1}(2Q^{-}+Q^{+})-R^{+}\Lambda_1\Lambda^{-1}(R^{+}-Q^{+})\equiv 0.\tag{A1.3}$$

(ii) *We consider the two other systems,* $M^{\pm}\psi=(-ik\Lambda+2R^{\pm}+Q^{-})\psi$, $M^{\pm}=L_{\rm I}\pm L_{\rm III}$, $L_{\rm III}=(\delta_{ij}\delta_i)$, where

$$\delta_1=\frac{\lambda_1}{\lambda_3}\frac{\partial}{\partial x_3}-\frac{\lambda_1}{\lambda_4}\frac{\partial}{\partial x_4},\qquad \delta_2=\frac{\lambda_2}{\delta_4}\frac{\partial}{\partial x_4}-\frac{\lambda_2}{\lambda_1}\frac{\partial}{\partial x_1},$$
$$\delta_3=\frac{\lambda_3}{\lambda_1}\frac{\partial}{\partial x_1}-\frac{\lambda_3}{\lambda_2}\frac{\partial}{\partial x_2},\qquad \delta_4=\frac{\lambda_4}{\lambda_2}\frac{\partial}{\partial x_2}-\frac{\lambda_4}{\lambda_3}\frac{\partial}{\partial x_3},\tag{A1.4}$$

and study the compatibility condition $[M^+,M^-]\psi=0$, which gives

$$0=-ik\Lambda(R^--R^+)\psi+M^-(R^+\psi)-M^+(R^-\psi)-L_{\rm III}(Q^-\psi)=0,\tag{A1.5}$$

with

$$L_{\rm III}(Q^-\psi)=(L_{\rm III}Q^-)\psi+Q^-\Lambda_4\Lambda^{-1}(Q^+-Q^-)\psi,$$
$$M^-(R^+\psi)=[(M^-R^+)+R^+\Lambda_1\Lambda^{-1}(-ik\Lambda+Q^++2R^+)]\psi,$$
$$M^+(R^-\psi)=[(M^+R^-)+R^-\Lambda_2\Lambda^{-1}(-ik\Lambda+R^++2Q^-)]\psi.$$

Taking these relations into account in (A1.5), the RHS becomes a sum of two terms. The first term, $-ik[\Lambda R^--R^-\Lambda_2-\Lambda R^++R^+\Lambda_1]$ is identically zero, so that the compatibility condition between the two systems is

$$M^-R^+-M^+R^--L_{\rm III}Q^-+R^+\Lambda_1\Lambda^{-1}(Q^++2R^+)$$
$$-R^-\Lambda_2\Lambda^{-1}(2Q^-+R^+)-Q^-\Lambda_4\Lambda^{-1}(Q^+-Q^-)\equiv 0.\tag{A1.6}$$

(iii) *Written with scalar quantities,* (A1.3) and (A1.6) give the two sets of nonlinear equations that the q_j^i must satisfy and which are written down in (6*a*).

Appendix 2

We consider formalism b and $n=3$.

We start with system (1), $(L_{\rm I}+ik\Lambda-Q_{\rm I})\psi=0$, where

$$L_{\rm I}=\left(\delta_{ij}\frac{\partial}{\partial x_i}\right),\qquad \Lambda=\begin{pmatrix}\lambda_1 & & 0\\ & \lambda_2 & \\ 0 & & \lambda_3\end{pmatrix},\qquad Q_{\rm I}=\begin{pmatrix}0 & q_2^1 & q_3^1\\ q_1^2 & 0 & q_3^2\\ q_1^3 & q_2^3 & 0\end{pmatrix},$$

and report the IE (4*b*) associated to (1):

$$K_j^i(x_1,x_2,x_3;y)=\tilde F_j^i(x_1,x_2,x_3;y)+\sum_m\int_{x_m}F_m^i(s;x_1,x_2,x_3;y)K_j^m(x_1,x_2,x_3;s)\,ds$$
$$\tilde F_j^i=F_j^i(s=x_j)F_j^i=F_j^i(x_j-s+\bar\lambda_i^jx_i+\bar\lambda_k^jx_k;\,x_i-y+\bar\lambda_j^ix_j+\bar\lambda_k^ix_k).\tag{A2.1}$$

(i) *Constraints satisfied by the solutions* K_j^i. We remark that the operator

$$A_j^i=\left(\frac{1}{\bar\lambda_i^j}\frac{\partial}{\partial x_i}-\frac{1}{\bar\lambda_k^j}\frac{\partial}{\partial x_k}\right)+\frac{\partial}{\partial y}\left(\frac{1}{\bar\lambda_i^j}-\frac{\bar\lambda_k^i}{\bar\lambda_k^j}\right)$$

applied to F_j^i or $\tilde F_j^i$ gives zero. We apply this operator to both sides of the IE to obtain

$$A_j^iK_j^i=\frac{\bar\lambda_i^k}{\bar\lambda_i^j}\tilde F_k^i\hat K_j^k-\frac{\bar\lambda_k^i}{\bar\lambda_k^j}\tilde F_i^i\hat K_j^i+\int F_i^iA_j^iK_j^i+\int F_i^i\left(\frac{1}{\bar\lambda_i^j}\frac{\partial}{\partial x_i}-\frac{1}{\bar\lambda_k^j}\frac{\partial}{\partial x_k}\right)K_j^i+\int F_k^i(-A_j^kK_j^k).$$

We write (A2.9) for K_j^i, K_j^k and apply respectively

$$\left(\frac{1}{\bar\lambda_i^j}\frac{\partial}{\partial x_i}-\frac{1}{\bar\lambda_k^j}\frac{\partial}{\partial x_k}\right)\qquad\text{and}\qquad \frac{\partial}{\partial y}\left(\frac{1}{\bar\lambda_i^j}-\frac{\bar\lambda_k^i}{\bar\lambda_k^j}\right).$$

$$\begin{aligned}
&A_j^iK_j^i=\frac{\bar\lambda_k^i}{\bar\lambda_k^j}K_i^i\hat K_j^i+\frac{\bar\lambda_i^k}{\bar\lambda_i^j}K_k^i\hat K_j^k, \qquad i\neq j,\\
&\left(\frac{1}{\bar\lambda_i^j}\frac{\partial}{\partial x_i}-\frac{1}{\bar\lambda_k^j}\frac{\partial}{\partial x_k}\right)K_j^i=\frac{\bar\lambda_i^k}{\bar\lambda_i^j}K_k^i\hat K_j^k-\frac{\bar\lambda_k^i}{\bar\lambda_k^j}K_i^i\hat K_j^i.
\end{aligned}\tag{A2.2}$$

(A2.2) is the equivalent in formalism b of the relation (11a) in formalism a or (11bb) in the formalism bb. Recalling the nonlinear relations (3b), $(\bar\lambda_j^i)^{-1}(\partial/\partial x_j)+\partial/\partial y)K_j^i=K_i^i\hat K_j^i+\bar\lambda_j^k(\bar\lambda_j^i)^{-1}\hat K_j^kK_k^i$, we can eliminate the K_i^i between (A2.2) and (3b) and take the limit $y=x_i$, leading to an equation for $\hat K_j^i$. In this way we obtain the nonlinear equation (5b) of § 5.4 which must be satisfied by the $\hat K_j^i$ or by the q_j^i if we take into account $q_j^i=\bar\lambda_j^i\hat K_j^i$.

(ii) *IE associated to another system* (1′)

We shall see that the IE (4b) (here called (A2.1) is also the IE of another system (1′), $(L_{\rm II}+\mathrm{i}k\Omega-P)\psi=0$, where

$$L_{\rm II}=(l_i\delta_{ij}),\qquad \Omega=(\omega_i\delta_{ij}),\qquad l_i=\frac{1}{\bar\lambda_{i+1}^i}\frac{\partial}{\partial x_{i+1}}-\frac{1}{\bar\lambda_{i+2}^i}\frac{\partial}{\partial x_{i+2}},$$

$$\omega_i=\frac{\lambda_{i+1}^i}{\bar\lambda_{i+1}^i}-\frac{\lambda_{i+2}^i}{\bar\lambda_{i+2}^i},\qquad P=\begin{pmatrix}0&-p_1^2&p_1^3\\ p_2^1&0&-p_2^3\\ -p_3^1&p_3^2&0\end{pmatrix},$$

with $i+2=1$ if $i=2$, and $i+1=1$, $i+2=2$ if $i=3$.

We assume that (1′) also has the set of solutions ψ_j given by (2b) and where $u_j=u^0_{\lambda_j}(x_j)u^0_{\lambda_i^j}(x_i)u^0_{\lambda_k^j}(x_k)$. Substituting (2$b$) into (1′) we obtain

$$\begin{aligned}
&\int_{x_j}^{\infty}u^0_{\lambda_j}(y)\left[\left(\frac{1}{\bar\lambda_k^j}\frac{\partial}{\partial x_k}-\frac{1}{\bar\lambda_i^j}\frac{\partial}{\partial x_i}\right)K_j^i+p_j^kK_k^i-p_j^iK_i^i\right]\mathrm{d}y=0,\\
&\int_{x_j}^{\infty}u^0_{\lambda_j}(y)\left\{\left[\frac{1}{\bar\lambda_k^i}\frac{\partial}{\partial x_k}-\frac{1}{\bar\lambda_j^i}\frac{\partial}{\partial x_j}+\left(\frac{\bar\lambda_k^i}{\bar\lambda_k^i}-\frac{1}{\bar\lambda_j^i}\right)\frac{\partial}{\partial y}\right]K_i^i-p_i^i+p_i^kK_k^i\right\}\mathrm{d}y\\
&+u^0_{\lambda_j}(x_j)\left(-p_i^j+\frac{\bar\lambda_k^i}{\bar\lambda_k^i}\hat K_i^j\right)=0.
\end{aligned}\tag{A2.3}$$

From (A2.3) it follows that, if $p_j^i=(\bar\lambda_k^i/\bar\lambda_k^i)\hat K_j^i$, then the relations are satisfied if the K_j^i transforms satisfy the nonlinear relations (A2.2). In conclusion, (1) and (1′) have the same IE (4b), and the nonlinear three-wave equation (5b) represents their compatibility because it is obtained from a combination of relations (A2.2) and (3b).

(iii) *If we interpret one coordinate as time, then the nonlinear three-wave equation* (5b) *has confined solution in* R^2 *space at finite time*

Let us consider the kernel $\mathscr{F}$ as given by the example (15a) of the previous paper (Cornille 1978a), but with degenerate kernels $F_j^i=g_j^ih_j^i$ given by equation (4b) of the present formalism b. The solutions $\hat K_j^i$ are still given by (15b) of the previous paper;

however, (15*c*) of that paper is replaced by

$$g_j^i(\bar\lambda_i^j x_i + \bar\lambda_k^j x_k), \qquad h_j^i(\bar\lambda_j^i x_j + \bar\lambda_k^i x_k), \qquad A_{j,k}^i = \int_{-\infty}^{\bar\lambda_i k_{x_i} + \bar\lambda_j^k x_i} g_k^i(u) h_j^k(u)\, du.$$

We assume $g_j^i(u) \to 0$, $h_j^i(u) \to 0$ when $|u| \to \infty$.

(iv) *Compatibility conditions between systems* (1) *and* (1′)

Let us recall that the relation between $\hat{K}_j^i$, the solution of (A2.1) (for (4*b*), $q_j^i \in Q_{\mathrm{I}}$ and $p_j^i \in P$ is $p_j^i = q_j^i \bar\lambda_j^i (\bar\lambda_k^i \bar\lambda_j^i)^{-1}$, $q_j^i = \bar\lambda_j^i \hat{K}_j^i$. The compatibility condition $[L_{\mathrm{I}}, L_{\mathrm{II}}]\psi = 0$ leads to

$$0 = -\mathrm{i}k(\Lambda P - \Omega Q_{\mathrm{I}})\psi + L_{\mathrm{II}}(Q_{\mathrm{I}}\psi) - L_{\mathrm{I}}(P\psi). \tag{A2.4}$$

Let us define

$$A_1 = \begin{pmatrix} 1/\bar\lambda_1^3 & & 0 \\ & 1/\bar\lambda_2^1 & \\ 0 & & 1/\bar\lambda_3^2 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1/\bar\lambda_1^2 & & 0 \\ & 1/\bar\lambda_2^3 & \\ 0 & & 1/\bar\lambda_3^1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} \bar\lambda_2^1/\bar\lambda_2^3 & & 0 \\ & \bar\lambda_3^2/\bar\lambda_3^1 & \\ 0 & & \bar\lambda_1^3/\bar\lambda_1^2 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} \bar\lambda_3^1/\bar\lambda_3^2 & & 0 \\ & \bar\lambda_1^2/\bar\lambda_1^3 & \\ 0 & & \bar\lambda_2^3/\bar\lambda_2^1 \end{pmatrix},$$

$$P = P^+ - P^-, \qquad P^+ = \begin{pmatrix} 0 & 0 & P_1^3 \\ P_2^1 & 0 & 0 \\ 0 & P_3^2 & 0 \end{pmatrix}, \qquad P^- = \begin{pmatrix} 0 & P_1^2 & 0 \\ 0 & 0 & P_2^3 \\ P_3^1 & 0 & 0 \end{pmatrix},$$

$Q_{\mathrm{I}} = Q^+ + Q^-$, where the $Q^\pm$ are defined in § 5.1. With some algebra we can factorise ψ in (A2.4) to obtain

$$0 = -\mathrm{i}k[\Lambda P - \Omega Q_{\mathrm{I}} + (Q^- A_1 - Q^+ A_2)\Lambda - (Q^- A_3 + Q^+ A_4)\Omega]$$
$$+ [L_{\mathrm{II}} Q_{\mathrm{I}} - L_{\mathrm{I}} P + (Q^- A_1 - Q^+ A_2) Q_{\mathrm{I}} - (Q^- A_3 + Q^+ A_4) P]. \tag{A2.5}$$

The first square bracket is identically zero, whereas the second square bracket, written with scalar quantities, gives the nonlinear three-wave equation (5*b*).

Appendix 3

For (2*c*) and $n = 2$ we establish the IE associated to (1), where $q_i^i = 0$,

$$\left(\frac{\partial}{\partial x_i} + \mathrm{i}k\lambda_i\right) u_i - \sum_{l=1}^{2} q_i^l u_l = 0. \tag{1c}$$

We define $u_j(x_1, x_2) = u_{\lambda_j}^0(x_j) u_{\lambda_{j'}^j}^0(x_{j'})$, $j' \neq j$ and consider

$$\psi_j = \left(\delta_{ij} u_j(x_1, x_2) + \sum_{m=1}^{2} \int_{x_m}^{\infty} K_{i,m}^j(x_1, x_2; y) u_j(x_m = y, x_{m'})\, dy\right), \qquad m' \neq m. \tag{2c}$$

We assume the boundary conditions

$$\lim_{y\to\infty} K_{11}^2 u_{\lambda_1^2}^0 = 0, \qquad \lim_{y\to\infty} u_{\lambda_2}^0 K_{12}^2 = 0, \qquad \lim_{y\to\infty} u_{\lambda_2}^0 K_{22}^2 = 0,$$

$$\lim_{y\to\infty} u_{\lambda_1}^0 K_{11}^1 = 0, \qquad \lim_{y\to\infty} u_{\lambda_1}^0 K_{21}^1 = 0, \qquad \lim_{y\to\infty} u_{\lambda_2^1}^0 K_{22}^1 = 0,$$

and substitute $(2c)$ into $(1c)$. We obtain

$$\sum_{m=1}^{2}\int_{x_m} u_j(y=x_m, x_{m'})\left[\left(\frac{\partial}{\partial x_i}+\gamma^i_{i,m}\frac{\partial}{\partial y}\right)K^i_{l,m}-\sum_l q^l_i K^i_{l,m}\right]\mathrm{d}y$$
$$+u_j(x_1,x_2)\left\{-q^l_i+(1-\delta_{ij})\left[\hat{K}^i_{i,j}\bar{\lambda}^i_i+\hat{K}^i_{ii}\left(-1+\frac{\lambda_i}{\lambda^j_i}\right)\right]\right\}\equiv 0,$$

where $\hat{K}^j_{i,m}=K^j_{i,m}(y=x_m)$, $\gamma^i_{ii}=1$, $\gamma^i_{ij}=0$, $\gamma^i_{jj}=\lambda_j(\lambda^i_j)^{-1}$, $\gamma^i_{ji}=\bar{\lambda}^i_j$. This identity can be satisfied if

$$q^i_i=0,\qquad q^l_i=\hat{K}^l_{i,j}\bar{\lambda}^l_i+\hat{K}^l_{ii}\left(-1+\frac{\lambda_i}{\lambda^l_i}\right),$$
$$\left(\frac{\partial}{\partial x_i}+\gamma^l_{i,p}\frac{\partial}{\partial y}\right)K^i_{i,p}=\left[\hat{K}^l_{i,l}\bar{\lambda}^l_i+\hat{K}^l_{ii}\left(-1+\frac{\lambda_i}{\lambda^l_i}\right)\right]K^i_{l,p},\qquad l\neq i. \tag{3c}$$

Let us consider the IE and define kernels $F^i_{j,p}(s_1; s_2; x_1, x_2; y)$:

$$K^i_{j,p}(x_1,x_2;y)=\tilde{F}^i_{j,p}(x_1,x_2;y)+\sum_{m=1}^{2}\sum_{q=1}^{2}\int_{x_q}F^i_{,p,q}(s;x_1,x_2;y)K^m_{j,q}(x_1,x_2;s)\,\mathrm{d}s,$$

$$\tilde{F}^i_{j,p}=F^i_{j,p}(s_1=x_1, s_2=x_2),\qquad F^i_{j,p,q}=F^i_{j,p}(s_q=s, s_{q'}=x_{q'}),\qquad q'\neq q$$

$$\left(\frac{\partial}{\partial x_m}+\gamma^i_{m,p}\frac{\partial}{\partial y}+\bar{\lambda}^i_m\frac{\partial}{\partial s}\right)F^i_{j,p,j}=0,\qquad m=1,2$$

$$\left(\frac{\partial}{\partial x_j}+\gamma^i_{j,p}\frac{\partial}{\partial y}\right)\tilde{F}^i_{j,p}=0,\qquad \left(\frac{\partial}{\partial x_j}+\gamma^i_{j,p}\frac{\partial}{\partial y}\right)F^i_{j,p,j'}=0,\qquad \left(\frac{\partial}{\partial x_{j'}}+\gamma^i_{j',p}\frac{\partial}{\partial y}+\frac{\lambda_{j'}}{\lambda^i_{j'}}\frac{\partial}{\partial s}\right)F^i_{j,p,j'}$$
$$=0,\qquad j\neq j',$$

or

$$F^i_{j,p}=F^i_{j,p}(\lambda_j(x_j-s_j)+\lambda_{j'}x_{j'}-\lambda^i_{j'}s_{j'};\ \gamma^i_{1p}x_1-y+\gamma^i_{2p}x_2),\qquad j'\neq j.$$

If we assume

$$\lim_{s\to\infty}F^i_{m,p,q}(s,x_1,x_2;y)K^m_{j,q}(x_1,x_2;s)=0,$$

then one can show that the solutions $K^i_{j,p}$ of $(4c)$ satisfy the nonlinear equations $(3c)$. For the proof we apply $(\partial/\partial x_j+\gamma^i_{j,p}\partial/\partial y)$ to both sides of the IE $(4c)$, and taking into account the properties of the kernels $F^i_{j,p,q}$ we obtain

$$\left(\frac{\partial}{\partial x_j}+\gamma^i_{j,p}\frac{\partial}{\partial y}\right)K^i_{j,p}-\sum_m\sum_q\int_{x_q}F^i_{m,p,q}\left(\frac{\partial}{\partial x_j}+\gamma^m_{j,q}\frac{\partial}{\partial s}\right)K^m_{j,q}$$
$$=\tilde{F}^i_{j',q}\left[\hat{K}^{j'}_{j,j}\left(-1+\frac{\lambda_j}{\lambda^{j'}_j}\right)+\bar{\lambda}^{j'}_j\hat{K}^{j'}_{jj'}\right],\qquad j'\neq j.$$

If we compare with the IE $(4c)$ (when we assume that the solution exists and is unique), we see that the set $\{K^i_{j,p}\}$ verifies $(3c)$.

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